CONCERNING ESSENTIAL CONTINUA OF CONDENSATION*

BY

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A collection G will be said to be of class 1 if (1) it is an upper semi-continuous collection of mutually exclusive continua, and (2) with respect to its elements it is a continuum with no continuum of condensation.

The subcontinuum K of the continuum M is said to be an *essential continuum of condensation* of M provided it is true that (1) K is a continuum of condensation of M, and (2) if G is a collection of class 1 filling up M, then some continuum of G contains K.

THEOREM 1. *If the non-degenerate subcontinuum K of the compact continuum M has no continuum of condensation and for every collection G of class 1 filling up M there is a continuum of G that contains K, then K is a continuum of condensation of M.*

Proof. Suppose K is not a continuum of condensation of M. Then there exists a point set D which is a domain both with respect to M and to K. Since K has no continuum of condensation, D contains a free† segment with respect to K, that is to say, a segment t of some arc lying in K such that no point of t is a limit point of $K-t$, in other words, such that t is a domain with respect to K. The point set t is also a domain with respect to M. If $M-t$ is a continuum, let G denote the collection whose elements are $M-t$ and the points of t. If $M-t$ is not connected, it is the sum of two mutually exclusive continua g_A and g_B containing the endpoints A and B respectively of t. In this case, let G denote the collection whose elements are g_A, g_B and the points of t. In the first case G is a simple closed curve, and in the second case it is an arc with respect to its elements. In each case it is a collection of class 1 and no continuum of G contains K. This involves a contradiction.

THEOREM 2. *If K is a subcontinuum of the compact continuum M, and for every collection G of class 1 filling up M there is a continuum of G that contains K, then K contains a continuum of condensation of M.*

Proof. Suppose K contains no continuum of condensation of M. Then it

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† Cf. R. L. Moore, *Foundations of Point Set Theory*, Colloquium Publications of the American Mathematical Society, vol. 13, 1932, p. 144. This book will be referred to as P.S.T.

contains no continuum of condensation of itself. Hence, by Theorem 1, K is itself a continuum of condensation of M.

THEOREM 3. *If the arc-wise connected continuum K is a subset of the compact continuum M, and for every collection G of class 1 filling up M there is a continuum of G that contains K, then K contains a continuum of condensation of M of diameter equal to that of K.*

Theorem 3 is a consequence of Theorem 1 and the fact that no arc has a continuum of condensation. That it does not remain true on the omission of the stipulation that K be arc-wise connected may be seen from the following example.

EXAMPLE 1. Let A, B, C, D, and E denote five points lying on a straight line in the order $ABCDE$ and such that each of the straight line intervals AB, BC, CD, and DE is of length 1. Let M_1, M_2, and M_3 denote three indecomposable continua of diameter 1 containing the intervals BC, CD, and DE respectively and such that no proper subcontinuum of M_2 contains both C and D. Let M denote the point set which is the sum of the interval AB and the continua M_1, M_2, and M_3. If G is a collection of class 1 filling up M, some continuum of G contains the continuum $M_1+M_2+M_3$ and therefore is of diameter not less than 3. But there is no continuum of condensation of M of diameter as great as 2.

THEOREM 4. *If the compact continuum M has no essential continuum of condensation, then for every non-degenerate subcontinuum K of M there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is, with respect to its elements, either an arc or a simple closed curve and such that no continuum of the collection G contains K.*

Proof. By hypothesis and Theorem 2 there exists a collection H of class 1 filling up M and such that no continuum of H contains K. Let W denote the collection whose elements are the continua of the collection H that intersect K. There exists a free segment t with respect to H such that every element of t belongs to W. By an argument analogous to that used in a similar connection in the above proof of Theorem 1, it may be shown that there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that G is either an arc or a simple closed curve with respect to its elements and such that every element of t is an element of G. The continuum K is not a subset of any continuum of the collection G.

NOTATION. If G is a collection of point sets, the notation G^* will be used to denote the sum of all the point sets of the collection G.

THEOREM 5. *If M is a compact regular curve, n is a positive integer and, for each i less than or equal to n, A_i and B_i are points of M and A_{n+1} is a point of*

M and B_{n+1} is a subset of M and there exist collections H and K of class 1 *filling up M and such that* (1) *if i is not greater than n, no continuum of H contains both A_i and B_i, and* (2) *the continuum of K that contains A_{n+1} contains no point of B_{n+1}; then there exists a collection G of class* 1 *filling up M such that, if $i \leqq n+1$, no continuum of G intersects both A_i and B_i.*

Proof. For each i not greater than n let a_i denote the continuum of H containing A_i and let b_i denote the one that contains B_i. Let a_{n+1} denote the continuum of H that contains A_{n+1}. For each i less than or equal to n, either a_i or b_i is distinct from a_{n+1}. Let L denote the set of all continua x distinct from a_{n+1} such that, for some i not more than n, x is identical either with a_i or with b_i. There exists a finite set T of subsets of H such that (1) each element of T is a free segment with respect to H, that is to say, a segment t of elements of H such that no element of t is a limit element of $H-t$, (2) the set $H-T$ is the sum of two mutually separated sets W_1 and W_2 of elements of H such that W_1 contains L and W_2 contains a_{n+1}. There exists a set T' of points such that (1) each point of the set T' is an element of H belonging to some segment of the set T, (2) for each segment t of the set T there is only one point of T' which is an element of t. The set $H-T'$ is the sum of two mutually separated sets Q_1 and Q_2 of elements of H such that W_1 and W_2 are subsets of Q_1 and Q_2 respectively. For each point P of the point set $T'+Q_2^*$ let x_P denote the continuum of the collection K that contains P, and let g_P denote the component of $x_P \cdot (T'+Q_2^*)$ that contains P. Let G denote the collection whose elements are the elements of Q_1 and the continua g_P for all points P of $T'+Q_2^*$.

THEOREM 6. *If M is a compact regular curve, n is a positive integer and, for each i not greater than n, A_i and B_i are points of M and, for each such i, there exists a collection H_i of class* 1 *filling up M and such that no continuum of H_i contains both A_i and B_i, then there exists a collection G of class* 1 *filling up M such that, if $i \leqq n$, no continuum of G contains both A_i and B_i.*

Proof. Suppose m is a positive integer for which there exists a collection G_m of class 1 filling up M and such that, if $i \leqq m$, no continuum of G_m contains both A_i and B_i. If $m<n$ then, by hypothesis and Theorem 5, there exists a collection G_{m+1} of class 1 filling up M such that, if $i \leqq m+1$, no continuum of this collection contains both A_i and B_i. The conclusion of Theorem 6 follows by mathematical induction.

THEOREM 7. *If M is a compact regular curve and d is a positive number and for every collection G of class* 1 *filling up M there is some continuum of G of diameter greater than or equal to d, then M has an essential continuum of condensation of diameter d.*

Proof. Suppose n is a positive integer such that $1/n$ is $<d$. There exists a finite set K of points of M such that every subcontinuum of M of diameter equal to $1/(3n)$ contains at least one point of K. Suppose N is a subcontinuum of M of diameter d. There exist two points A and B of N at a distance apart equal to d. There exist two subcontinua a and b of N containing A and B respectively and each of diameter $1/3n$. The continua a and b contain points A' and B' belonging to K. The distance from A' to B' is greater than $d-1/n$. It follows that there exist finite sequences $A_1, A_2, \cdots, A_j$ and $B_1, B_2, \cdots, B_j$ such that (1) for each i less than j, A_i and B_i are points of K at a distance apart greater than $d-1/n$, and (2) if N is a subcontinuum of M of diameter d, then, for some i, A_i and B_i are points of N. Hence, by hypothesis and Theorem 6, there exists a number m_n such that if G is any collection of class 1 filling up M, then some continuum of the collection G contains both A_{m_n} and B_{m_n}. Let H denote the set of all continua h such that h contains both A_{m_n} and B_{m_n} and belongs to some collection of class 1 filling up M. Let T denote the common part of all the continua of the set H. Suppose T contains no continuum that contains both A_{m_n} and B_{m_n}. Then T is the sum of two mutually exclusive closed point sets T_A and T_B containing A_{m_n} and B_{m_n} respectively. There exists a finite point set L that separates T_A from T_B in M. Every continuum of the set H contains both A_{m_n} and B_{m_n} and therefore some point of L. Hence, by Theorem 6, there exists a point of L that belongs to every continuum of H and therefore to T. This is impossible. Hence T contains a continuum containing both A_{m_n} and B_{m_n}. Thus for every positive integer n there exists a subcontinuum K_n of M of diameter greater than $d-1/n$ and such that if G is any collection of class 1 filling up M, then some continuum of G contains K_n. There exists an ascending sequence of positive integers $n_1, n_2, n_3, \cdots$ such that the sequence $K_{n_1}, K_{n_2}, K_{n_3}, \cdots$ has a sequential limiting set E. The point set E is a continuous curve of diameter d. If G is any collection of class 1 filling up M, some continuum of G contains E. There exist two points X and Y belonging to E and at a distance apart equal to d. There exists an arc XY lying in E. The arc XY is an essential continuum of condensation of M of diameter d.

It is clear from Example 1 that Theorem 7 does not remain true if the requirement that M be a compact regular curve is replaced by the weaker requirement that it be a compact and webless* continuum.

* The compact continuum M is said to be *webless* provided it is true that if G_1 and G_2 are two upper semi-continuous collections of mutually exclusive continua filling up a subcontinuum of M and such that each of them is an acyclic continuous curve with respect to its elements, then there does not exist an uncountable subcollection H of G_1 such that no continuum of H is a subset of any continuum of G_2 (cf. P.S.T., p. 355).

THEOREM 8. *If M is a compact and webless continuum and there exists a positive number d such that, if G is any collection of class* 1 *filling up M, some continuum of G is of diameter more than d, then M has an essential continuum of condensation.*

Proof. Suppose M has no essential continuum of condensation. Then, by Theorem 36 of page 361 of P.S.T. together with Theorem 2 of the present paper, M is a regular curve. Hence, by Theorem 7, it has an essential continuum of condensation of diameter d.

In P.S.T., a subcontinuum K of a compact continuum M was called an essential continuum of condensation of M of type 1 if there exists a non-degenerate subcontinuum T of K such that, if G is a collection of class 1 filling up M, then some continuum of G contains T; and the subcontinuum K of M was called an essential continuum of condensation of M of type 2 if there exist a point P of K and a point O of M distinct from P such that, if G is a collection of class 1 filling up M, then O belongs to the continuum of G that contains P. The first of these definitions is open to the objection* that an essential continuum of condensation of type 1 of a continuum M is not necessarily a continuum of condensation of M. However, by Theorem 2, every essential continuum of condensation of type 1 (in the sense of P.S.T.) of a compact continuum M *contains* a continuum which is a continuum of condensation of M and which is therefore an essential continuum of condensation of M in the sense of the present treatment; and if an essential continuum of condensation of type 1 of the compact continuum M is itself a continuum with no continuum of condensation, then it *is* an essential continuum of condensation of M in the present sense.

It is easy to see that there exists a compact continuum M containing a continuum K which is an essential continuum of condensation of M of type 2 in the sense of P.S.T. but which does not contain more than one point of any continuum of condensation of M. However, by Theorem 8, if the compact and webless continuum M has an "essential continuum of condensation of type 2," then it has an essential continuum of condensation in the present sense.

THEOREM 9. *If M is a compact continuum such that every non-degenerate subcontinuum of M contains uncountably many local separating points† of M,*

* Cf. R. L. Moore, *Fundamental Theorems of Point Set Theory*, The Rice Institute Pamphlet, vol. XXIII (1936), p. 58. The word "non-degenerate" is to be omitted in the statement of Axiom 2 on p. 3.

† A point P is a *local separating point* of M provided some neighborhood V of P exists such that $M\cdot\bar{V}-P$ is separated between some pair of points belonging to the component of $M\cdot\bar{V}$ which contains P. See G. T. Whyburn, *Local separating points of continua*, Monatshefte für Mathematik und Physik, vol. 36 (1929), p. 305, and *Sets of local separating points of a continuum*, Bulletin of the American Mathematical Society, vol. 39 (1933), p. 97.

and H and K are two mutually exclusive closed subsets of M, then there do not exist infinitely many mutually exclusive subcontinua of M each containing both a point of H and a point of K.

Proof. Suppose there exists an infinite sequence α of mutually exclusive subcontinua of M each intersecting both H and K. There exists a subsequence β of α converging to some continuum L that intersects both H and K. There exists a non-degenerate subcontinuum T of L containing no point of $H+K$. By hypothesis L contains an uncountable set Q of local separating points of M belonging to T. By a theorem of Whyburn's,* there exists a point P of Q of Menger order† 2 of M. There exists a domain D with respect to M containing P such that $\bar{D}$ contains no point of H or of K and such that γ, the boundary of D with respect to M, consists of only two points. Since P belongs to the sequential limiting set of β, infinitely many continua of β contain points of D. But each of them contains points belonging to $H+K$ and therefore to $S-D$. Hence each of them intersects γ. But the continua of β are mutually exclusive and γ is a finite point set. This involves a contradiction.

THEOREM 10. *If AB is an arc lying in the compact continuum M and containing uncountably many local separating points of M, then there exists an open subset D of M containing AB such that there are uncountably many points each separating A from B in $\bar{D}$.*

Proof. By the above mentioned theorem of Whyburn's there exists an uncountable subset K of $AB-(A+B)$ such that every point of K is of Menger order 2 of M. Suppose P is a point of K. There exists an open subset I_P of M containing P such that (1) $\bar{I}_P$ contains neither A nor B, (2) the boundary of I_P consists of two points, (3) I_P-P is the sum of two connected and mutually separated point sets containing A_P and B_P respectively of the boundary, with respect to M, of I_P such that A_P lies between A and P, and B_P lies between B and P, on the arc AB. There exist open subsets H_{AP} and H_{BP} of M containing AA_P and BB_P respectively and such that $\bar{H}_{AP}$ and $\bar{H}_{BP}$ are mutually exclusive. The point set $H_{AP}+I_P+H_{BP}$ is an open subset T_P of M containing AB, and P separates A from B in T_P.

There exists a sequence $D_1, D_2, D_3, \cdots$ of open subsets of M closing down on AB. For each point P of K there exists a number n_P such that $\bar{D}_{n_P}$ is a subset of T_P. The point set K is uncountable. Hence there exist a number m and an uncountable subset L of K such that, if P is any point whatsoever

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* G. T. Whyburn, *Concerning the cut points of continua*, these Transactions, vol. 30 (1928), p. 606.

† Cf. K. Menger, *Grundzuge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925), pp. 272-306.

of L, $n_P = m$. For every point P of L, $\overline{D}_m$ is a subset of T_P and therefore P separates A from B in D_m.

THEOREM 11. *If every non-degenerate subcontinuum of the compact continuum M contains uncountably many local separating points of M, then M has no essential continuum of condensation.*

Proof. Suppose K is a non-degenerate subcontinuum of M. By Theorem 9, and Theorem 8 of Chapter 2 of P.S.T., every subcontinuum of M is a continuous curve. Hence K contains an arc AB. By Theorem 10 there exists a domain D, with respect to M, containing AB and such that there are uncountably many points each separating A from B in D. Let N denote the set of all points that separate A from B in D. The point set N contains a perfect set.* Hence it contains a perfect point set H having no non-degenerate component. Let Q denote the set of all subcontinua q of AB such that (1) q does not contain more than two points of H, but (2) every subcontinuum of AB of which q is a proper subset contains more than two points of H. Let W denote the set of all components x of $M-AB$ such that there are at least two continua of the set Q each containing a limit point of x. Each point set of the set W contains at least one point of $M-D$. Hence, by Theorem 9, W is a finite set. Furthermore if x is a point set of the set W, there do not exist infinitely many continua of the set Q each containing a limit point of x. Hence if L denotes the set of all continua q of Q such that q contains a limit point of some point set of the set W, the set L is finite. Hence there exist two points E and F such that (1) E and F are both continua of the set Q, and (2) the interval EF of AB contains no limit point of any point set of the set W. Let T denote the set of all continua of the set Q which are subsets of EF. For each continuum t of the set T, let g_t denote the component of $M-(EF-t)$ that contains t. Let G_1 denote the set of all continua g_t for all continua t of the set T except E and F. If $\bar{g}_E+\bar{g}_F$ is not a continuum, let G denote the collection whose elements are g_E, g_F and the elements of G_1. If $\bar{g}_E+\bar{g}_F$ is a continuum C, let G denote the collection whose elements are C and the elements of G_1. In the first case G is an arc, and in the second case it is a simple closed curve, with respect to its elements. In neither case does any continuum of G contain the continuum K. Hence M has no essential continuum of condensation.

THEOREM 12. *If the compact and webless continuum M has no essential continuum of condensation, then every non-degenerate subcontinuum of M contains uncountably many local separating points of M.*

Proof. Suppose K is a non-degenerate subcontinuum of M. By hypothesis

* G. T. Whyburn, *Cut points of connected sets and of continua*, these Transactions, vol. 32 (1930), p. 151.

and Theorem 2 there exists a collection G of class 1 filling up M and such that no continuum of the collection G contains K. Let H denote the collection of all the continua of G that intersect K. There exists a free segment t with respect to G whose elements belong to H. But there exists no uncountable set of mutually exclusive non-degenerate subcontinua of M. Hence uncountably many of the elements of t are points of K. But clearly every point of K which is an element of t is a local separating point of M.

Theorem 12 does not remain true on the omission from its hypothesis of the requirement that M be webless. For a plane continuum consisting of a square together with its interior has no essential continuum of condensation and no one of its points separates it locally.

Neither is it true that if the subcontinuum K of the compact and webless continuum M contains no essential continuum of condensation of M, then K contains uncountably many local separating points of M. Consider the following example.

EXAMPLE. In a Cartesian plane S let OX and OY denote the axes of coordinates. If h is a positive number and E and F are points of OX such that x_E, the abscissa of E, is less than x_F, the abscissa of F, then by the *R-set of altitude h whose base is EF* will be meant the continuous curve M described in Example 2 on page 347 of P.S.T. except that A, B, A', and B' denote (x_E, h), (x_F, h), E, and F, respectively, instead of $(0, 1)$, $(1, 1)$, $(0, 0)$, and the point which was erroneously labelled $(0,1)$ instead of $(1,0)$ in the first sentence of the description of Example 1 on page 346. In accordance with this new meaning of the letters A, B, A', and B', $A_1'B_1'$, $A_2'B_2'$, $A_3'B_3'$, $\cdots$ of course denote subintervals of EF. For each n, the interval $A_n'B_n'$ will be called the nth Cantor interval associated with the R-set of altitude h whose base is EF.

Let M_1 denote the R-set of altitude 1 whose base is the interval whose endpoints are $(0, 0)$ and $(1, 0)$. For each n, let r_{1n} denote the nth Cantor interval associated with M_1, and let M_{1n} denote the R-set of altitude $1/(n+2)$ whose base is r_{1n}. Let W_1 denote the collection of R-sets M_{11}, M_{12}, M_{13}, $\cdots$. For each m, let M_{1nm} denote the R-set of altitude $1/(n+2)(m+2)$ whose base is the mth Cantor interval associated with M_{1n}. Let W_2 denote the collection of all R-sets M_{1nm} for all sensed pairs of positive integers n and m. This process may be continued indefinitely. Thus there exists an infinite sequence W_1, W_2, W_3, $\cdots$ such that (1) for each k, W_k is the collection of all R-sets $M_{1n_1n_2\cdots n_k}$ for all sensed sets of k positive integers n_1, n_2, $\cdots$, n_k, (2) for each k, and sensed set of k positive integers n_1, n_2, n_3, $\cdots$, n_k, $M_{1n_1n_2n_3\cdots n_kn_{k+1}}$ is the R-set of altitude $1/(n_1+2)(n_2+2)\cdots(n_{k+1}+2)$ whose base is the (n_k+1)th Cantor interval associated with $M_{1n_1n_2\cdots n_k}$, (3) for each n, M_{1n} is as described above.

Let M denote the point set which is the sum of M_1 and all the R-sets of the collections W_1, W_2, W_3, $\cdots$. Let K denote the straight line interval whose extremities are (0, 0) and (1, 0). The continuum K contains no local separating point of M. Nevertheless, though it is a continuum of condensation of M, it contains no essential continuum of condensation of M.

THEOREM 13. *If M is a compact continuum and W is a set of collections of class* 1 *each filling up M, and G is the collection of all continua g such that, for some point P of M, g is the component containing P of the common part of all continua x such that x contains P and belongs to some collection of the set W, then G is itself a collection of class* 1.

Proof. The collection G is an upper semi-continuous collection of mutually exclusive continua. Suppose there exists a continuum K of elements of G which is an essential continuum of condensation of G. There exists a collection Q of the set W such that no continuum of the collection Q contains every continuum of the set K. For each continuum q of the collection Q let t_q denote the set of all continua of G which are subsets of q. Each set t_q is a continuum of elements of G and the collection of all sets t_q for all continua q of the collection Q is an upper semi-continuous collection T of mutually exclusive continua of elements of G. But, with respect to its elements, T is a continuum with no continuum of condensation and K is not a subset of any set of the collection T. This involves a contradiction. Hence G belongs to class 1.

With the help of Theorem 13 the following theorem may be easily established.

THEOREM 14. *If Q is the collection of all continua that have no essential continua of condensation, then every compact continuum has Q-atomic* subsets.*

DEFINITION.† The continuum M is said to have ***property** N* if for every positive number e there exists a finite set G of mutually exclusive non-degenerate subcontinua of M such that every subcontinuum of M of diameter greater than e contains some continuum of the set G.

THEOREM 15. *If a compact continuum has property N, it is a regular curve.*

Proof. Suppose M is a compact continuum with property N. It is clear

* If M is a continuum and Q is a topological collection of continua, then the subcontinuum K of M is said to be a *Q-atomic subset* of M provided there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that, with respect to its elements, G belongs to Q and such that (1) K is an element of G and (2) if H is any other upper semi-continuous collection of mutually exclusive continua filling up M such that H is, with respect to its elements, a continuum of the collection Q, then every continuum of the collection G is a subset of some continuum of the collection H, or in other words, each continuum of H is either an element of G or the sum of two or more elements of G. (Cf. The Rice Institute Pamphlet, loc. cit.)

† The Rice Institute Pamphlet, loc. cit.

that there do not exist a positive number d and infinitely many mutually exclusive subcontinua of M all of diameter greater than d. Hence M is a continuous curve. Suppose A and B are two distinct points of M. There exists a finite set H of mutually exclusive non-degenerate subcontinua of M such that every subcontinuum of M of diameter greater than one half the distance from A to B contains some continuum of the set H. There exists a finite point set T such that each continuum of the set H contains only one point of T and each point of T belongs to some continuum of the set H and is distinct from A and from B. Every subcontinuum of M that contains A and B contains some continuum of the set H and therefore some point of T. Hence, since M is a continuous curve, T separates A from B in M. Therefore M is a regular curve.

THEOREM 16. *In order that the regular curve M should fail to have property N it is necessary and sufficient that it should contain an arc AB such that every segment of AB contains an endpoint of an arc in M which has only its endpoints in common with AB.*

Proof. That this condition is sufficient was shown in The Rice Institute Pamphlet. The necessity will now be shown. Suppose the regular curve M does not have property N. Then there exists a positive number e such that if H is any finite set of mutually exclusive non-degenerate subcontinua of M, there exists a subcontinuum of M with diameter more than e but containing no continuum of the set H. There exists, in M, an arc A_1B_1 of diameter greater than e. There exists a finite set H_1 of mutually exclusive intervals of A_1B_1 such that every interval of A_1B_1 of diameter more than 1 contains an arc of the set H_1. There exists in M an arc A_2B_2 of diameter more than e and containing no arc of the set H_1. There exists a set H_2 of mutually exclusive intervals of A_2B_2 such that every interval of A_2B_2 of diameter more than $1/2$ contains an arc of the set H_2. Let K denote the set of all arcs of the set H_2 which are not subsets of any arc of the set H_1. For each arc k of the set K there exists an interval h_k of k intersecting no arc of the set H_1. Let K' denote the set of all arcs h_k for all arcs k of the set K, and let T_1 denote the set of all arcs of H_1 which contain no arc of H_2. Let H_2' denote the set $T_1+(H_2-K)+K'$. The set H_2' is a finite set of mutually exclusive arcs. Hence there exists in M an arc A_3B_3 of diameter more than e and containing no arc of the set H_2'. The arc A_3B_3 contains no arc either of the set H_1 or of the set H_2. This process may be continued indefinitely. It follows that there exist two infinite sequences $A_1B_1, A_2B_2, \cdots$ and $H_1, H_2, \cdots$ such that, for each n, (1) A_nB_n is an arc, of diameter more than e, lying in M, (2) H_n is a finite set of mutually exclusive intervals of A_nB_n such that every interval of A_nB_n of length more than $1/n$ contains an interval of the set H_n, (3) the arc $A_{n+1}B_{n+1}$ contains no interval of any one of the sets $H_1, H_2, \cdots, H_n$. There exists an infinite sequence of positive integers $n_1, n_2, n_3, \cdots$ such that the sequence $A_{n_1}B_{n_1}, A_{n_2}B_{n_2}, \cdots$ has a non-degenerate sequential limiting set K. Since M is a regular curve, the continuum K is a regular curve and therefore it contains an arc AB. Suppose E and F are two points of AB. There exist points E_1, E_2, E_3, E_4, E_5, and E_6 lying on AB in the order $EE_1E_2E_3E_4E_5E_6F$. There exist two mutually exclusive connected point sets D and R containing E_3 and E_6 respectively and such that (1) D and R are both domains with respect to M, (2) $D\cdot(AB)$ is a subset of the segment of AB whose endpoints are E_2 and E_4 and $R\cdot(AB)$ is a subset of the one whose endpoints are E_5 and F. There exists a number i such that $A_{n_i}B_{n_i}$ contains neither of the intervals E_1E_2 and E_4E_5 of AB but contains points X and Y belonging to D and R respectively and distinct from E_3 and E_6 respectively. There exist arcs XE_3 and YE_6 lying in D and R respectively. The continuum $XE_3+A_{n_i}B_{n_i}+YE_6$ contains an arc E_3ZE_6 with endpoints at E_3 and E_6. There exist points O_1 and O_2 lying respectively on the segments E_1E_2 and E_4E_5 of AB and such that E_3ZE_6 contains neither O_1 nor O_2. Let W denote the common part of E_3ZE_6 and the point set which is the sum of the intervals AO_1 and O_2B of AB. The point set W is closed and contains E_6. Hence there is a first point of the set W in the order from E_3 to E_6 on E_3ZE_6. Let L denote this point, and let Y denote the last point of the interval O_1O_2 of AB in the order from E_3 to L on the interval E_3L of E_3ZE_6. The interval YL of E_3ZE_6 is an arc in M with one endpoint on the segment EF of AB and having only its endpoints in common with AB.

THEOREM 17. *If AB is an arc and, for each positive integer n, H_n is a finite set of points $X_{1n}, X_{2n}, \cdots, X_{m_nn}$ lying on AB in the order indicated, X_{1n} and X_{m_nn} being A and B respectively, and H_n being a subset of H_{n+1}, and Q_n is a set of arcs $X_{1n}Z_{1n}X_{2n}, X_{2n}Z_{2n}X_{3n}, \cdots, X_{m_n-1,n}Z_{m_n-1,n}X_{m_nn}$ each having only its endpoints in common with AB, and every point of AB is a limit point of the point set obtained by adding together all the arcs of the sets $Q_1, Q_2, Q_3, \cdots$ and M is a regular curve containing that point set, then AB is an essential continuum of condensation of M.*

Theorem 17 may be established by an argument similar to that employed on pages 61 and 62 of vol. XXIII of The Rice Institute Pamphlet to prove that a certain straight line interval AB is an essential continuum of condensation of a certain continuum M.

THEOREM 18. *If the compact and webless continuum M does not have property N, there exists an upper semi-continuous collection G of mutually exclusive continua filling up M such that, with respect to its elements, G is a continuum with an essential continuum of condensation.*

Proof. Suppose M has no essential continuum of condensation. Then, by Theorem 36 of Chapter V of P.S.T., it is a regular curve. Hence, by Theorem 16 of this paper and Theorem 7 of page 67 of vol. XXIII of The Rice Institute Pamphlet, there exists an arc AB lying in M such that if t is any segment of AB and K is a closed subset of $M-t$, there is an arc lying in M and having in common with AB only its endpoints which belong to t. It easily follows that there exist two infinite sequences $H_1, H_2, H_3, \cdots$ and $Q_1, Q_2, Q_3, \cdots$ such that, for each n, (1) H_n is a set of 2^n points $X_{1n}, X_{2n}, \cdots, X_{2^n,n}$ lying on AB in the order indicated, (2) for each odd number i less than 2^n there are only four points of the set H_{n+1} between the points X_{in} and $X_{i+1,n}$ of the set H_n, (3) Q_n is a set of arcs $X_{1n}Z_{1n}X_{2n}, X_{3n}Z_{2n}X_{4n}, X_{5n}Z_{3n}X_{6n}, \cdots, X_{2^n-1,n}Z_{2^n-1,n}X_{2^n,n}$ each lying in M and having only its endpoints in common with AB and each of diameter less than $1/n$. For each n, let I_n denote the set of all points X such that X lies on AB between the endpoints of some arc of the set H_n. The set I_n is the sum of 2^{n-1} segments of AB and I_{n+1} is a subset of I_n. Let T denote the common part of all the point sets of the infinite sequence $I_1, I_2, I_3, \cdots$. The point set T is closed, totally disconnected, and uncountable. For each point P of $AB-T$, let k_P denote the component of $AB-T$ that contains P. Let K denote the set of all the point sets $\bar{k}_P$ for all points P of $AB-T$. Let G denote the collection whose elements are the continua of the collection K and the points of M that belong to no continuum of the collection K. Let L denote the collection whose elements are the continua of G which are subsets of the arc AB and, for each n and i $(i \leqq 2^n)$, let X'_{in} denote the continuum of the collection G that contains X_{in}. For each n and odd number i less than 2^n, let $X'_{in}Z_{in}X'_{i+1,n}$ denote the set whose elements are X'_{in}, $X'_{i+1,n}$ and the points of $X_{in}Z_{in}X_{i+1,n}$ which do not belong to AB. For each n, let H_n' denote the collection whose elements are $X_{1n}', X_{2n}', \cdots, X'_{2^n,n}$ and let Q_n' denote the collection whose elements are $X_{1n}'Z_{1n}X_{2n}', X_{3n}'Z_{2n}X_{4n}', \cdots, X'_{2^n-1,n}Z_{2^n-1,n}X'_{2^n,n}$. Let A' and B' denote the continua of G that contain A and B respectively. In the space whose "points" are the elements of G, (1) L is an arc $A'B'$ with endpoints at A' and B', (2) for each n, H_n' is a set of points $X_{1n}', X_{2n}', \cdots, X'_{2^n,n}$ lying on $A'B'$ in the order indicated, X_{1n}' and $X'_{2^n,n}$ being A' and B' respectively, (3) H_n' is a subset of H'_{n+1}, (4) Q_n' is a collection of arcs each having only its endpoints in common with $A'B'$ and, finally, G is a regular curve and each point of $A'B'$ is a limit point of the point set $H_1'+H_2'+H_3'+\cdots$. It follows, by Theorem 17, that $A'B'$ is an essential continuum of condensation of G.

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